An improved chiral expansion using a pion-mass dispersion relation

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SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

THE UNIVERSITY OF ADELAIDE AUSTRALIA

CoEPP ARC Centre of Excellence for Particle Physics at the Terascale

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- The pion-mass dispersion relation
  - Example: the nucleon mass
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- Adding in finite-range regularization (FRR)
- Baryon chiral perturbation theory (BχPT)
- Confronting the data: chiral extrapolation
- Conclusion

Aims

- to understand the chiral behaviour of hadrons, and obtain a quantitative description of chiral symmetry breaking.
- to improve relativistically upon the properties of the heavy-baryon expansion, leading to baryon chiral perturbation theory (BχPT).
- to import the method of finite-range regularization (FRR) without compromising any symmetries, whilst inheriting its advantageous features.
- to perform a more reliable chiral extrapolation of lattice QCD results, i.e. reducing the systematic uncertainty.

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Introduction

What is a dispersion relation?

- A Kramers-Kronig dispersion relation tells us about the analyticity of a complex function.

- Recall that an analytic function may be written $f = u + iv$ (u, v: real-valued on some domain Ω). The real part of f, (i.e. u) can be defined in terms of its harmonic conjugate (v) via a Hilbert transform:

$$u(t_0) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}t \frac{v(t)}{t - t_0}, \quad t_0, t \in \Omega.$$

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Why the pion mass?

- In χPT, the quark mass, m_q, is not a fixed parameter. We expand about the 'chiral limit', $m_q \to 0$, to obtain chiral formulae.
- Observables such as nucleon mass ($f \equiv M_N$) or anomalous magnetic moment (AMM) ($f \equiv \kappa$) become functions of m_q.
- The quark mass is related to the pion-mass squared by the Gell-Mann–Oakes–Renner Relation (GOR): $m_q \propto m_\pi^2$.
- Looking at the complex plane of m_π^2, the observables, f, are analytic- except for a branch-cut in the negative real-axis, associated with pion-production.

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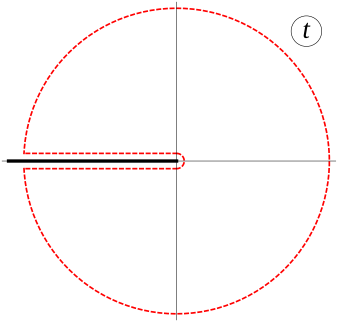

Figure: The complex $t = m_\pi^2$ plane, with the **branch-cut** along the negative real axis, and the contour indicating the analyticity domain.

A general pion-mass dispersion relation

- The pion-mass dispersion relation takes the general form (for static quantity, f):

$$\mathrm{Re}\, f(m_\pi^2) = -\frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}t \frac{\mathrm{Im}\, f(t)}{t - m_\pi^2}.$$

- We can analyze the analytic properties of the observables, f, very easily:
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The pion-mass dispersion relation

Loop-momentum vs. pion-mass

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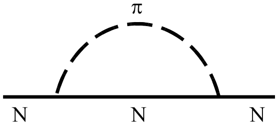
Loop momentum vs. pion mass

- The leading-order 1-pion loop takes the following simplified form in heavy-baryon χPT (HBχPT):



$$= \Sigma^{\mathrm{HB}}_{\pi N}(m_\pi^2) = \chi \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, \frac{k^4}{k^2 + m_\pi^2}.$$

- After a change of integration variable, $t = -k^2$, we get:

$$\Sigma^{\mathrm{HB}}_{\pi N}(m_\pi^2) = -\frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}t \, \frac{\chi \, (-t)^{3/2}}{t - m_\pi^2} \quad \leftarrow$$

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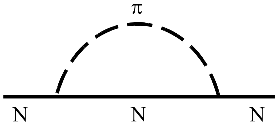

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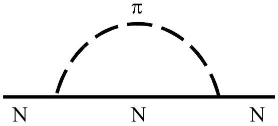

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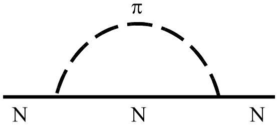

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- This formula is simply the dispersion relation, with $\mathrm{Im}\,\Sigma^{\mathrm{HB}}_{\pi N}(t) = \chi\,(-t)^{3/2}$, and t taking values on the negative real-axis branch-cut in the complex plane.
- The dispersion relation is satisfied, since the nonanalytic term, χm_π^3, from the chiral formula is the only contributor to $\mathrm{Im}\,\Sigma^{\mathrm{HB}}_{\pi N}(m_\pi^2)$ at order m_π^3.
- We shall use the t-integration form of the loop integral from now on, for two main reasons:
  - It is explicitly clear that no symmetries of the theory are violated, even if a (sharp) finite-range cutoff in the t-integral is introduced.
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Subtractions

- We would like to write the chiral formula for $M_N(m_\pi^2)$ in terms of our new dispersion relation.
- $\Sigma_{\pi N}^{\rm HB}(m_\pi^2)$ gives us the correct nonanalytic contribution to M_N, but also gives extra analytic terms:

$$\Sigma_{\pi N}^{\rm HB}(m_\pi^2) = b_0 + b_2 m_\pi^2 + \chi m_\pi^3.$$

- The extra terms b_0 and $b_2 m_\pi^2$ should be absorbed (renormalized) into $\overset{\circ}{M}_N$ and $c_2 m_\pi^2$, respectively. Thus, we must subtract off the extra terms from the loop integral $\Sigma_{\pi N}^{\rm HB}$. (Use C.I.F. for derivatives):

$$\Sigma_{\pi N}^{\rm HB}(m_\pi^2) - \Sigma_{\pi N}^{\rm HB}(0) - \Sigma_{\pi N}^{\rm HB}{}'(0) m_\pi^2 = -\frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}t \frac{\mathrm{Im}\,\Sigma_{\pi N}^{\rm HB}(t)}{t - m_\pi^2}\left(\frac{m_\pi^2}{t}\right)^2.$$

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- The chiral expansion formula for M_N can now be written in terms of the subtracted dispersion relation:

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- In general, we write the dispersion relation for n subtractions as:

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- The chiral expansion formula for M_N can now be written in terms of the subtracted dispersion relation:

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Finite-range regularization

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- Finite-range regularization (FRR) has some useful properties:
  - It is suitable for composite particles, where degrees of freedom at higher energy scales exist (quarks/gluons).
  - It can be used to determine the power-counting regime (PCR) of χPT, where the chiral expansion is convergent [hep-lat/0501028].
  - It can be used to improve the heavy-baryon expansion by resumming the chiral series so the higher-order terms are small [hep-lat/0302020].
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- Recall: chiral formulae are not convergent series expansions in general, but are asymptotic expansions in m_π/Λ_χ, ($\Lambda_\chi \simeq 4\pi f_\pi \sim 1$ GeV).
- Consider the dispersion relation integral split into two parts:

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- The first integral contains the chiral nonanalytic terms, but has been cut off at the scale: $-\Lambda^2$.
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- Since we are working to chiral order m_π^3, we only have a finite number of low-energy coefficients. But the terms from the second integral should be comparable to other higher-order terms neglected.
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- We would like to use this FRR dispersion relation to compare results from heavy-baryon and non-heavy-baryon chiral perturbation theory (HBχPT vs. BχPT).
- By imposing the cutoff, $\Lambda \sim 1$ GeV, we can investigate the convergence properties of the chiral expansion in m_π^2, without computing the higher order terms (often done in the FRR literature).
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Baryon chiral perturbation theory

Properties at $\mathcal{O}(m_\pi^3)$: nucleon mass

- Recall the chiral expansion formula for the nucleon mass M_N:

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$${\rm Im}\, M_N(t) \overset{\rm HB}{=} {\rm Im}\,\{\chi\, t^{3/2}\} = \chi\, t\sqrt{-t}\,\theta(-t).$$

- In BχPT, one obtains the following formula from the covariant integral result (for physical nucleon mass scale $\hat{M}_N \simeq 939$ MeV):

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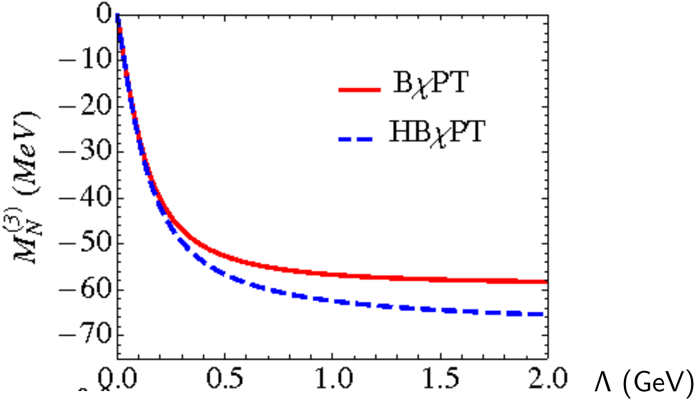

Figure: The Λ-dependence of leading-order loop contributions to the nucleon mass, $M_N^{(3)} \equiv \tilde{\Sigma}_{\pi N}$, calculated in HB$\chi$PT (blue dashed curves) and BχPT (red solid curves) at $m_\pi^2 = m_{\pi,\text{phys}}^2$.

Properties at $\mathcal{O}(m_\pi^3)$: nucleon mass

- The HBχPT formula can be obtained from the BχPT by taking the heavy-baryon limit: $\hat{M}_N \to \infty$.
- We will find that, for all our examples of f, the HBχPT formulae contain the term:

$$\tilde{\Sigma}_f^{\mathrm{HB}}(m_\pi^2;\Lambda) = -2\chi m_\pi^{2n-1} \arctan\frac{\Lambda}{m_\pi}.$$

- n (the number of subtractions) specifies the leading nonanalytic term (in m_π^2).
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Properties at $\mathcal{O}(m_\pi^3)$: AMMs

- A similar treatment follows for the anomalous magnetic moment (AMM), κ, of the proton and neutron.
- The finite-size behaviour of a hadron (pion-cloud corrections) leads to an anomalous component, κ, to its magnetic moment (in addition to its Dirac moment).
- The leading-order contribution to the AMM is a 1-pion loop with minimal insertion of one photon.
- The imaginary parts of the AMMs in HBχPT are:

$$\mathrm{Im}\,\kappa_p(t) \stackrel{\mathrm{HB}}{=} -\frac{4}{3}\chi \hat{M}_N \sqrt{-t}\,\theta(-t) = -\mathrm{Im}\,\kappa_n(t).$$

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Properties at $\mathcal{O}(m_\pi^3)$: AMMs

- Chiral formulae corresponding to HBχPT and BχPT may again be obtained by evaluating the dispersion relation:

$$\kappa_{p,n}^{\mathrm{FRR}} = \overset{\circ}{\kappa}_{p,n} - \frac{1}{\pi}\int_{-\Lambda^2}^{0} \mathrm{d}t \frac{\mathrm{Im}\,\kappa_{p,n}(t)}{t - m_\pi^2}\left(\frac{m_\pi^2}{t}\right)^1 .$$

- In lattice QCD, the isovector nucleon $(p - n)$ is used, so that calculations involving all-to-all propagators cancel.
- The isovector nucleon AMM formula may also include a term linear in m_π^2 if desired (for fitting):

$$\kappa_{\mathrm{isov}}(m_\pi^2;\Lambda^2) = \kappa_p(m_\pi^2;\Lambda^2) - \kappa_n(m_\pi^2;\Lambda^2) + a_2 m_\pi^2 .$$

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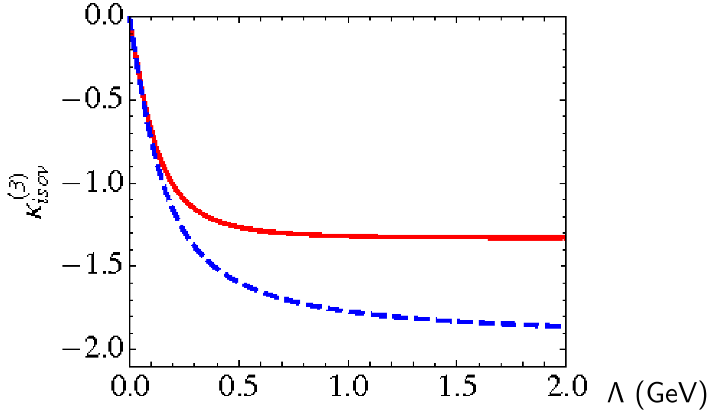

Figure: The Λ-dependence of leading-order loop contributions to the isovector nucleon AMM, calculated in HBχPT (blue dashed curves) and BχPT (red solid curves) at $m_\pi^2 = m_{\pi,\mathrm{phys}}^2$.

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- We shall also consider the magnetic polarizability, β_p, of the proton.
- Its leading-order contribution is a 1-pion loop with minimal insertion of two photons.
- The imaginary part of the polarizability in HBχPT is:

$$\operatorname{Im}\beta_p(t) \overset{\text{HB}}{=} -\frac{\alpha}{18}\chi\frac{1}{\sqrt{-t}}\theta(-t) \quad (\alpha \simeq 1/137).$$

- Here, the leading-order nonanalytic term is $\sim 1/m_\pi$. No subtractions are required. Furthermore, this negative power of m_π^2 will have consequences for the heavy-baryon expansion.

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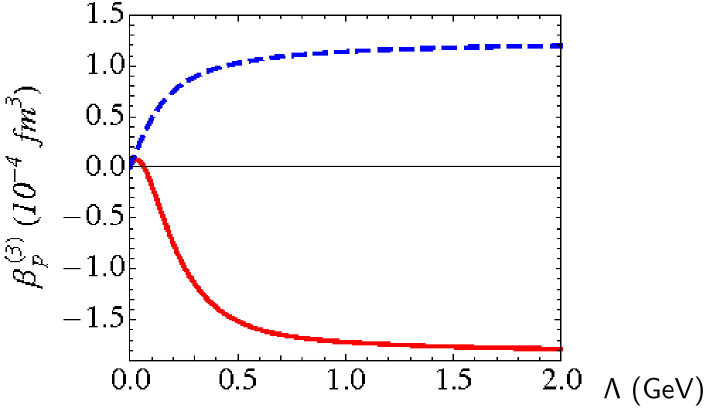

Figure: The Λ-dependence of leading-order loop contributions to the proton magnetic polarizability, calculated in HBχPT (blue dashed curves) and BχPT (red solid curves) at $m_\pi^2 = m_{\pi,\mathrm{phys}}^2$.

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- The residual Λ-dependence in HBχPT falls off as $1/\Lambda$ in all examples, whereas in BχPT, it behaves as $1/\Lambda^2$ for M_N, and $1/\Lambda^4$ for the AMMs and polarizability.
- The stronger dependence on Λ indicates a greater impact from the unknown high-energy physics to be renormalized.
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- In the case of the magnetic polarizability of the proton, there is significant difference in the results, even at $\Lambda \sim m_{\pi,\mathrm{physical}} \ll 1$ GeV, and the results are the opposite sign!
- This is because the BχPT formula contains contributions $\sim -1/\hat{M}_N$, which are largely underestimated in HBχPT.
- Recalling the formula:

$$\tilde{\Sigma}_f{}^{\mathrm{HB}}(m_\pi^2;\Lambda) = -2\chi m_\pi^{2n-1} \arctan\frac{\Lambda}{m_\pi},$$

the power index, n, allows us to classify the naturalness of the heavy-baryon expansion. The lower the value of n, the greater the difficulty for HBχPT to describe a quantity.

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Confronting the data: chiral extrapolation

Confronting the data

- The results of χPT must be matched to an underlying theory.
- In the case of polarizabilities, there are no unknown parameters at leading order, so a χPT result is a genuine prediction.
  But: there are currently no lattice results to use, and the experimental value is uncertain.
- For the nucleon mass, we don't expect much difference between HBχPT and BχPT near the physical pion mass, but the difference can be significant for larger pion masses.
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M_N extrapolation- lattice QCD

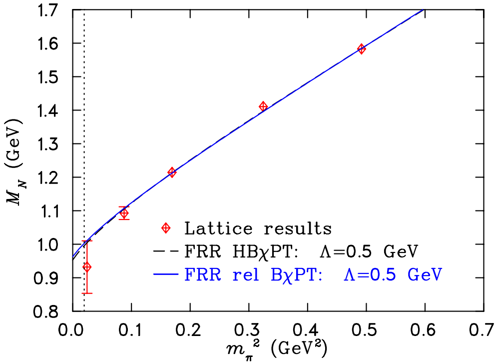

Figure: Chiral extrapolations of the nucleon mass for HBχPT compared to BχPT at $\Lambda = 0.5$ GeV. The extrapolation based on PACS-CS results, box size: 2.9 fm. Finite-volume effects are neglected.

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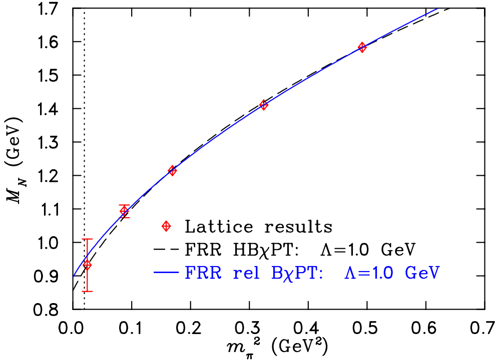

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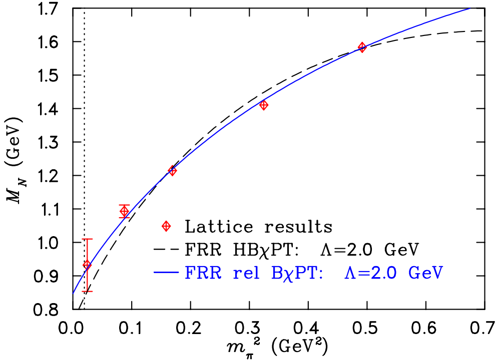

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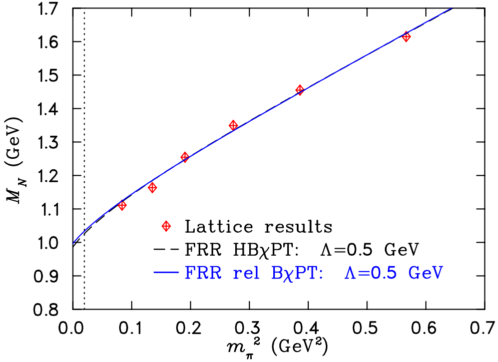

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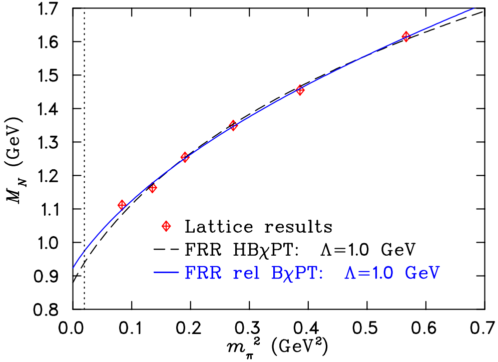

Figure: Chiral extrapolations of the nucleon mass for HBχPT compared to BχPT at $\Lambda = 1.0$ GeV. The extrapolation based on JLQCD results, box size: 1.9 fm. Finite-volume effects are neglected.

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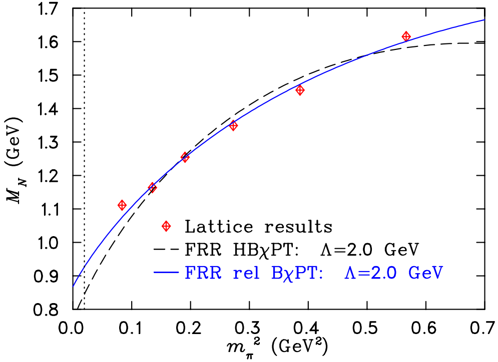

Figure: Chiral extrapolations of the nucleon mass for HBχPT compared to BχPT at $\Lambda = 2.0$ GeV. The extrapolation based on JLQCD results, box size: 1.9 fm. Finite-volume effects are neglected.

M_N extrapolation- lattice QCD

- For small values of Λ, the chiral loops are suppressed, an almost-linear fit ensues, yielding a poor fit to the low pion-mass lattice results.
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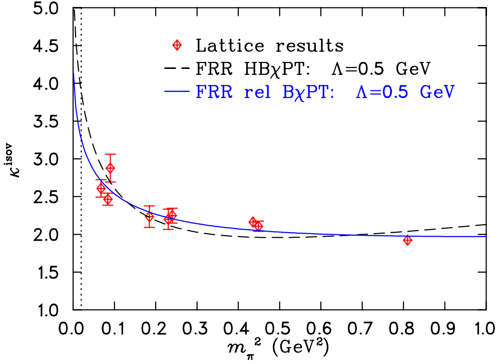


Figure: Chiral extrapolations of the isovector nucleon AMM for HBχPT compared to BχPT at $\Lambda = 0.5$ GeV. The extrapolation based on QCDSF results, box size: $1.7 - 2.9$ fm. Finite-volume effects are neglected.

AMM extrapolation- lattice QCD

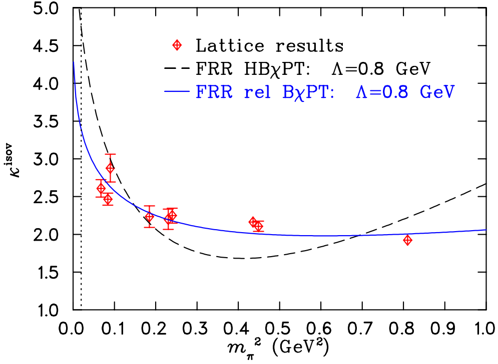
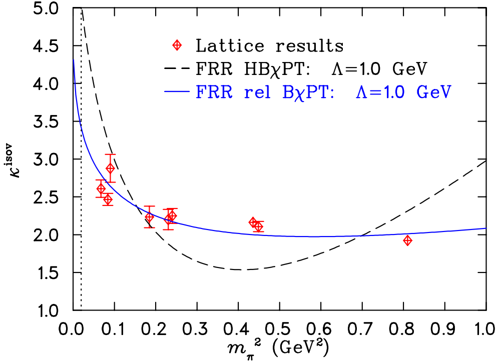


Figure: Chiral extrapolations of the isovector nucleon AMM for HBχPT compared to BχPT at $\Lambda = 0.8$ GeV. The extrapolation based on QCDSF results, box size: $1.7 - 2.9$ fm. Finite-volume effects are neglected.

AMM extrapolation- lattice QCD



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AMM extrapolation- lattice QCD

- In the AMM extrapolation, we see larger chiral curvature than the case of M_N, because of its lower-order leading nonanalytic term ($\sim m_\pi$).
- For this reason, the HBχPT extrapolation becomes unfavorable at large values of Λ, with large curvature for $\Lambda \gtrsim 1$ GeV.
- Even with the inclusion of the linear 'a_2 term', which plays the role of compensating for high-momentum contributions, the BχPT result is much more stable to changes in ultraviolet behaviour.

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- We incorporated the useful properties of finite-range regularization into our chiral expansion formulae.
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Helpful references

- Limitations of the heavy-baryon expansion: J. M. M. Hall and V. Pascalutsa, arXiv:1203.0724 [hep-ph].
- The pion-mass dispersion relation: T. Ledwig, V. Pascalutsa & M. Vanderhaeghen, Phys.Lett. **B690**, 129 (2010), 1004.3449.
- ‘Naturalness’: H. Georgi, Nucl.Phys. **B361**, 339 (1991).
- The magnetic polarizability of the proton: V. Lensky and V. Pascalutsa, Eur.Phys.J. **C65**, 195 (2010), 0907.0451.
- Nucleon mass lattice QCD results:
  - S. Aoki *et al.*, (PACS-CS Collaboration), Phys.Rev. **D79**, 034503 (2009), 0807.1661.
  - H. Ohki *et al.*, Phys.Rev. **D78**, 054502 (2008), 0806.4744.
- AMM lattice QCD results: S. Collins, M. Gockeler, P. Hagler, R. Horsley, Y. Nakamura, J. Zanotti *et al.*,Phys.Rev. **D84**, 074507, (2011), 1106.3580.

Appendix

FRR BχPT chiral formulae

- The relativistically-improved chiral formula for κ_p is:

$$\kappa_p^{\mathrm{FRR}} \stackrel{\mathrm{B}}{=} \overset{\circ}{\kappa}_p$$

$$+ \frac{2\chi}{3\pi} \hat{M}_N^2 \Bigg\{ \frac{m_\pi\left(-8 + 22\frac{m_\pi}{\hat{M}_N^2} - 6\frac{m_\pi^4}{\hat{M}_N^4}\right)}{\hat{M}_N \sqrt{4 - \frac{m_\pi^2}{\hat{M}_N^2}}} \arctan\left(\frac{\Lambda}{m_\pi}\sqrt{\frac{4\hat{M}_N^2 - m_\pi^2}{4\hat{M}_N^2 + \Lambda^2}}\right)$$

$$- \frac{m_\pi^2}{\hat{M}_N^2}\left(5 - \frac{3m_\pi^2}{\hat{M}_N^2}\right)\left[2\,\mathrm{arcsinh}\frac{\Lambda}{2\hat{M}_N} + \log\frac{m_\pi^2}{m_\pi^2 + \Lambda^2}\right]$$

$$+ \frac{3m_\pi^2\Lambda^2}{\hat{M}_N^4}\left(1 - \sqrt{1 + \frac{4\hat{M}_N^2}{\Lambda^2}}\right)\Bigg\}.$$

FRR BχPT chiral formulae

- The relativistically-improved chiral formula for κ_n is:

$$\kappa_n = \mathring{\kappa}_n + \frac{8\chi}{3\pi} \hat{M}_N^2 \Bigg\{ \frac{m_\pi (2 - \frac{m_\pi^2}{\hat{M}_N^2})}{\hat{M}_N \left(4 - \frac{m_\pi^2}{\hat{M}_N^2}\right)^{1/2}} \arctan \left(\frac{\Lambda}{m_\pi} \sqrt{\frac{4\hat{M}_N^2 - m_\pi^2}{4\hat{M}_N^2 + \Lambda^2}} \right)$$

$$+ \frac{m_\pi^2}{2\hat{M}_N^2} \left[2 \operatorname{arcsinh} \frac{\Lambda}{2\hat{M}_N} + \log \frac{m_\pi^2}{m_\pi^2 + \Lambda^2} \right] \Bigg\}.$$

FRR BχPT chiral formulae

$$\beta_p = \frac{2\alpha\,\chi}{9\pi}\left\{ \frac{2(2 - 246\frac{m_\pi^2}{\hat{M}_N^2} + 471\frac{m_\pi^4}{\hat{M}_N^4} - 212\frac{m_\pi^6}{\hat{M}_N^6} + 27\frac{m_\pi^8}{\hat{M}_N^8})}{m_\pi\left(4 - \frac{m_\pi^2}{\hat{M}_N^2}\right)^{3/2}} \right.$$

$$\times \arctan\left(\frac{\Lambda}{m_\pi}\sqrt{\frac{4\hat{M}_N^2 - m_\pi^2}{4\hat{M}_N^2 + \Lambda^2}}\right) - \left(\frac{9}{\hat{M}_N} - \frac{50 m_\pi^2}{\hat{M}_N^3} + \frac{27 m_\pi^4}{\hat{M}_N^5}\right)$$

$$\times \left[2\,\mathrm{arcsinh}\frac{\Lambda}{2\hat{M}_N} + \log\frac{m_\pi^2}{m_\pi^2 + \Lambda^2}\right] - \frac{\Lambda^2}{\hat{M}_N^3}\left[\frac{27(\Lambda^2 - 2m_\pi^2)}{2\hat{M}_N^2}\right.$$

$$\times \left(1 - \sqrt{1 + \frac{4\hat{M}_N^2}{\Lambda^2}}\right) + 50 - 23\sqrt{1 + \frac{4\hat{M}_N^2}{\Lambda^2}}$$

$$\left.\left. - \frac{51\hat{M}_N^6}{\Lambda^2(4\hat{M}_N^2 + \Lambda^2)(4\hat{M}_N^2 - m_\pi^2)}\right]\right\}.$$