Chiral Effective Field Theory Beyond the Power Counting Regime

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Overview

- **Introduction**
- **Effective Field Theory for nucleons**
  - Loop integrals
  - Renormalization
- **Ideal ‘pseudodata’**
- **Intrinsic energy scale**
  - Evidence
  - Statistical uncertainty
  - Higher chiral order
- **Quenched $\rho$ meson case**
- **Conclusion & future directions**
Lattice QCD can rarely be evaluated at physical quark masses. We want to be able to extrapolate current results to this physical point.

Chiral Perturbation Theory gives insight into this low energy region, but is limited to use over a very small range of quark masses.

We will discover that using more of the available data often entails model-dependence. But the extent of the model-dependence can be quantified and thus removed.

This will lead us to realizing the presence of an ‘intrinsic energy scale’, embedded in such lattice QCD data.
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Why use $\chi$EFT?

- **Chiral Effective Field Theory ($\chi$EFT)** complements lattice QCD.
  - It assists in understanding the consequences of dynamical chiral symmetry breaking.
  - It provides a scheme-independent approach for investigating the properties of hadrons.
  - In particular, it can be used in conjunction with lattice QCD data to extrapolate results:
    - to physical quark masses,
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- $\chi$PT provides a formal expansion in terms of low energy momenta and quark masses.
- The expansion is convergent if the quark mass is small so that higher order terms are negligible. This is called the **Power Counting Regime (PCR)**.
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Formal Background

- For an effective field theory, one writes out a **low energy effective Lagrangian**.
  - The terms of the Lagrangian are ordered in powers of momenta and mass.
  - For nucleons (fermions) written as an SU(2) doublet $\Psi = (p \ n)^T$, the first order (lowest energy) Lagrangian takes the form:

\[
\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi} \left( \phi - \circ M_N + \frac{g_A}{2f_\pi} \gamma^\mu \gamma_5 \vec{\tau} \cdot \partial_\mu \vec{\pi} \right) \Psi,
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- The circle $\circ$ denotes a “bare” quantity: it gets renormalized by chiral loops from the field theory. Let’s look at the nucleon mass $M_N$ ...
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The nucleon mass $M_N$ is renormalized by:

- an analytic polynomial associated with the quark masses $m_q$.
- chiral loop integrals $\Sigma_{\text{loops}}$.

The low energy expansion formula about the chiral limit (small $m_q$) is expressed using the Gell-Mann—Oakes—Renner Relation $m_q \propto m_\pi^2$:

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M_N = \{\text{terms analytic in } m_\pi^2\} + \{\text{chiral loop corrections}\}
= \{a_0 + a_2 m_\pi^2 + a_4 m_\pi^4 + O(m_\pi^6)\} + \{\Sigma_{\text{loops}}\}.
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The analytic terms will be collectively called the ‘residual series’, and their coefficients $a_i$ will be determined by fitting to lattice QCD data.

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The integral form of the chiral loops are obtained using the Feynman Rules for $\chi$PT, and can then be solved.

Each loop, when evaluated from its integral form, produces a non-analytic term.

To finite chiral order ($O(m_\pi^4 \log m_\pi)$), the leading order chiral loops are:

- the 1–pion loop ($\Sigma_N \sim m_\pi^3$),
- the pion loop decuplet transition ($\Sigma_\Delta \sim m_\pi^4 \log m_\pi$),
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+ \left( -\frac{3}{4\pi\Delta} \chi_\Delta + \chi'_t \right) m_\pi^4 \log \frac{m_\pi}{\mu} + \mathcal{O}(m_\pi^5).
\]
Renormalization Issues

- The coefficients $\chi_N$, $\chi_\Delta$ & $\chi_t'$ are known, scheme-independent parameters (related to $g_A$, $f_\pi$, etc).
- The coefficients $b_i$ however, are scheme-dependent, but they occur at the relevant chiral orders to renormalize the residual series:

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How does the renormalization take place? Consider the 1–pion loop integral as a test example:

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\Sigma_N = \frac{2\chi_N}{\pi} \int_{0}^{\infty} \frac{k^4}{k^2 + m_{\pi}^2} \, dk
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\Sigma_N(\Lambda) = \frac{2\chi_N}{\pi} \int_0^\Lambda dk \frac{k^4}{k^2 + m_\pi^2}
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The massless renormalization scheme result is recovered as $\Lambda \to \infty$.

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Taking the heavy-baryon limit and performing the $k_0$ integration, the loop integrals take the following forms:

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\tilde{\Sigma}_N = \frac{\chi_N}{2\pi^2} \int d^3k \frac{k^2 u^2(k; \Lambda)}{\omega^2(k)} - b_{0,N}^\Lambda - b_{2,N}^\Lambda m_\pi^2,
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\tilde{\Sigma}_\Delta = \frac{\chi_\Delta}{2\pi^2} \int d^3k \frac{k^2 u^2(k; \Lambda)}{\omega(k)(\Delta + \omega(k))} - b_{0,\Delta}^\Lambda - b_{2,\Delta}^\Lambda m_\pi^2,
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\[
\tilde{\Sigma}_{tad} = c_2 m_\pi^2 \left( \frac{\chi_t}{4\pi} \int d^3k \frac{2u^2(k; \Lambda)}{\omega(k)} - b_{2,t}^\Lambda \right)
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= c_2 m_\pi^2 \tilde{\sigma}_{tad}, \quad \text{pulling out } c_2 \text{ as a factor}.
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The pion energy is $\omega(k) \equiv \sqrt{k^2 + m_\pi^2}$ and the FRR regulator function is denoted by $u(k; \Lambda)$. 
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Note that the tadpole integral has a coefficient $\chi_t' = c_2 \chi_t$, which involves $c_2$ (obtained from the Lagrangian $\mathcal{L}^{(2), tad}_{\pi N} = c_2 \text{Tr}_f [\mathcal{M}_q] \bar{\Psi} \Psi$).

Thus the nucleon mass expansion formula can be conveniently factorized:

$$M_N = \left\{ a_0 + a_2 m_\pi^2 + a_4 m_\pi^4 + O(m_\pi^6) \right\} + \left\{ \Sigma_N + \Sigma_\Delta + \Sigma_{tad} \right\} = c_0 + c_2 m_\pi^2 (1 + \tilde{\sigma}_{tad}) + a_4^\Lambda m_\pi^4 + \tilde{\Sigma}_N + \tilde{\Sigma}_\Delta.$$

This formula can be used for extrapolations, with fit coefficients $c_0$, $c_2$ and $a_4^\Lambda$. 
Expansion Formulae

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Expansion Formulae

- Note that the tadpole integral has a coefficient $\chi'_t = c_2 \chi_t$, which involves $c_2$ (obtained from the Lagrangian $\mathcal{L}^{(2),tad}_{\pi N} = c_2 \text{Tr}_f [\mathcal{M}_q] \bar{\Psi}\Psi$).

- Thus the nucleon mass expansion formula can be conveniently factorized:

$$M_N = \{a_0 + a_2 m^2_\pi + a_4 m^4_\pi + \mathcal{O}(m^6_\pi)\} + \{\Sigma_N + \Sigma_\Delta + \Sigma_{tad}\}$$

$$= c_0 + c_2 m^2_\pi (1 + \tilde{\sigma}_{tad}) + a^\Lambda_4 m^4_\pi + \tilde{\Sigma}_N + \tilde{\Sigma}_\Delta.$$

- This formula can be used for extrapolations, with fit coefficients $c_0$, $c_2$ and $a^\Lambda_4$. 

Finite Volume Corrections

- **Lattice QCD is done on a finite volume box.**
- Our ideal infinite volume expansion formula should be modified to include finite volume corrections.
- Each integral can be converted into a discrete summation, and then the difference is taken to achieve the correction:

\[
\delta_{i}^{FVC} = \frac{\chi i}{2\pi^2} \left[ \frac{(2\pi)^3}{L_x L_y L_z} \sum_{k_x, k_y, k_z} - \int d^3k \right].
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- The tadpole finite volume corrections are subtle and will not be dealt with in this talk.
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Finite Volume Corrections

- The **finite volume corrections** are easily incorporated into our expansion formula:

$$M_N^V = c_0 + c_2 m_\pi^2 (1 + \tilde{\sigma}_{tad}) + a_4^\Lambda m_\pi^4$$

$$+ (\tilde{\Sigma}_N + \delta_{N}^{FVC}) + (\tilde{\Sigma}_\Delta + \delta_{\Delta}^{FVC}) + O(m_\pi^5).$$

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- We are almost ready to try an extrapolation from lattice QCD data. But what form ought the regulator \( u(k; \Lambda) \) to take?
Finite-Range Regulators

- All forms of \( u(k; \Lambda) \) are equivalent within the PCR, as long as they are normalized to 1, and are suppressed to 0 for large momenta \( k \). Dimensional Regularization (DR) corresponds to \( \Lambda \to \infty \).
- The step function \( \theta(\Lambda - k) \) is acceptable, but is unfavorable for use with the finite volume of the lattice.
- Consider the family of smooth \( n \)-tuple dipole attenuators:
  \[
  u_n(k; \Lambda) = \left(1 + \frac{k^{2n}}{\Lambda^{2n}}\right)^{-2}.
  \]
  - The dipole corresponds to \( n = 1 \). We shall also consider the cases \( n = 2, 3 \), the double and triple dipole forms, respectively.
  - We shall analyze data using these three different regulators to demonstrate the model-independence of this approach.
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- We shall analyze data using these three different regulators to demonstrate the model-independence of this approach.
Here are the three dipole-like forms plotted for $\Lambda = 1.0$ GeV:
Consider the behaviour of $M_N$ as a function of $m_{\pi}^2$.

Here is an extrapolation of data from JLQCD, using a dipole regulator with $\Lambda_{\text{dip}} = 1.0$ GeV.

The JLQCD data uses $N_f = 2$ overlap fermions at $L = 1.9$ fm.

![Graph showing the behaviour of $M_N$ as a function of $m_{\pi}^2$.](attachment://graph.png)
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Here is an extrapolation of data from PACS-CS, using a dipole regulator with $\Lambda_{\text{dip}} = 1.0$ GeV.

The PACS-CS data uses non-perturbatively $O(a)$-improved Wilson quark action at $L = 2.9$ fm.
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![Graph showing $M_N$ as a function of $m_{\pi}^2$]

- $+$ original data
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Here is an extrapolation of data from CP-PACS, using a dipole regulator with $\Lambda_{\text{dip}} = 1.0 \text{ GeV}$.

The CP-PACS data uses a mean field improved clover quark action at $L = 2.2 \rightarrow 2.8 \text{ fm}$.
Trial Extrapolations

- Consider the behaviour of $M_N$ as a function of $m_{\pi}^2$.
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- Consider the behaviour of $M_N$ as a function of $m_\pi^2$.
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Trial Extrapolations

- There is nothing special about $\Lambda_{\text{dip}} = 1.0$ GeV.
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What happens to the extrapolation as $\Lambda_{dip}$ is changed? Consider the CP-PACS data:
Different choices of regulator give different results! But is there an optimal choice?

Also, if we want to stay close to the PCR, how many data points should we use? Does it matter?

Let’s do a test: Using the extrapolation formula for $M_N$, generate some ideal ‘pseudodata’.

- Generate one set of 100 closely spaced, low energy pseudodata points entirely within the PCR, created at $\Lambda_{\text{dip}}^c = 1.0$ GeV.

- Generate two more sets, at different upper values $m_{\pi,\text{max}}^2$, thus progressing outside the PCR.

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We can use these pseudodata sets for our analysis of regulator dependence.

The regulator dependence is characterized by the behaviour of the renormalized constants $c_i$ with respect to $\Lambda_{\text{dip}}$.

Let’s plot our fit coefficients $c_0$ and $c_2$ over a range of $\Lambda_{\text{dip}}$ values, for each of the three data sets. We have chosen $m_{\pi,\text{max}}^2 = 0.04, 0.25, 0.5 \text{ GeV}^2$. 
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The PACS-CS data, on which the pseudodata is based, is shown below:

- $\frac{M_N}{(\text{GeV})}$
- $m_\pi^2 (\text{GeV}^2)$

- + original data
- $\Lambda = 1.0 \text{ GeV, dipole, fin.vol.}$
- $\Lambda = 1.0 \text{ GeV, dipole, inf.vol.}$
Here is the result for $c_0$.

Notice that the correct value of $c_0$ is recovered exactly when $\Lambda_{\text{dip}} = \Lambda_{c_{\text{dip}}}$.
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This intersection point is not trivial. To demonstrate this, we can analyze the pseudodata using a triple dipole. Here is the result for $c_0$: 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing the dependence of $c_0$ on $\Lambda$ (GeV) for different values of $m_{\pi,max}^2$.}
\end{figure}
Here is the result for $c_2$:

This intersection is no longer a clear point, but a cluster at $\Lambda_{dip} \approx 0.5 - 0.6$ GeV. This is the preferred value of $\Lambda_{trip}$. 
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\[ c_2 \]

\[ \Lambda \]
- The regulator dependence increased as the pseudodata extended outside the PCR.
- We also see that FRR breaks down if \( \Lambda \) is too small.
- This makes sense mathematically, as \( b_i^\Lambda \propto \Lambda^{3-i} \), and so for \( i = 4, 6, \ldots \) these higher order coefficients blow up for small \( \Lambda \).
- This also makes sense physically, as any ultraviolet regulator \( \Lambda \) must be large enough to allow inclusion of the chiral physics being studied. Otherwise we essentially destroy the non-analytic behaviour by making the integrals \( \approx 0 \).
- Thus there is a lowest suitable value \( \Lambda_{\text{lower}} \) below which we cannot ensure consistent renormalization.
Pseudodata- Renormalization Flow

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Evidence for an Intrinsic Scale

- In the pseudodata test example, the optimal cutoff (by construction) was obtained from the pseudodata themselves.
  
  - But do actual lattice QCD data have an intrinsic scale embedded in them?
  
  - If so, it would indicate that the data contain information regarding an optimal FRR regulator.
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Evidence for an Intrinsic Scale

- Let us repeat our analysis of $c_0$ and $c_2$ for the JLQCD, PACS-CS and CP-PACS data sets.

- We will obtain each one using the lightest 4 data points, and increase $m_{\pi,\text{max}}^2$ by one data point at a time.

- Each time we add a new data point, we increase the distance the data set extends outside the PCR, thus increasing the scheme-dependence. This helps identify the intrinsic scale.

- Since actual lattice QCD data is not ideal like our pseudodata, we can’t expect that the renormalization flow curves will cross at exactly the same value of $\Lambda$. 
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Evidence for an Intrinsic Scale

Here is the result for $c_0$ using JLQCD data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a dipole regulator:

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\centering
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- $m_{\pi,\text{max}}^2 = 0.27$ GeV$^2$
- $m_{\pi,\text{max}}^2 = 0.39$ GeV$^2$
- $m_{\pi,\text{max}}^2 = 0.57$ GeV$^2$
Evidence for an Intrinsic Scale

Here is the result for $c_0$ using JLQCD data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a double dipole regulator:

\[
\begin{align*}
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Evidence for an Intrinsic Scale

- Here is the result for $c_0$ using JLQCD data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a triple dipole regulator:

![Graph showing the behavior of $c_0$ as a function of $\Lambda$ for different $m_{\pi,\text{max}}^2$ values.]

- $m_{\pi,\text{max}}^2 = 0.27$ GeV$^2$
- $m_{\pi,\text{max}}^2 = 0.39$ GeV$^2$
- $m_{\pi,\text{max}}^2 = 0.57$ GeV$^2$
Evidence for an Intrinsic Scale

Here is the result for $c_2$ using JLQCD data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a dipole regulator:

\[
\begin{align*}
\text{Graph showing } c^\alpha \text{ vs. } \Lambda \text{ (GeV)} \text{ with different lines for different } m_{\pi,\text{max}}^2 \text{ values.}
\end{align*}
\]
Evidence for an Intrinsic Scale

Here is the result for $c_2$ using JLQCD data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a double dipole regulator:

\[ c^\alpha \]

\[ \Lambda \text{ (GeV)} \]

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\begin{align*}
\text{Evidence for an Intrinsic Scale} \\
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\end{align*}
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Here is the result for $c_0$ using PACS-CS data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a dipole regulator:

$$m_{\pi,\text{max}}^2 = 0.32 \text{ GeV}^2$$

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$$\Lambda (\text{GeV})$$

$$c^\alpha$$

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![Graph showing the relationship between $c_2$ and $\Lambda$ with two lines, one for $m_{\pi,\text{max}}^2 = 0.32$ GeV$^2$ and another for $m_{\pi,\text{max}}^2 = 0.49$ GeV$^2$.](image-url)
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![Graph showing $c^\Lambda$ vs $\Lambda$ with two curves, one for $m_{\pi,\text{max}}^2 = 0.32$ GeV$^2$ and another for $m_{\pi,\text{max}}^2 = 0.49$ GeV$^2$.](image-url)
Evidence for an Intrinsic Scale

- Here is the result for $c_0$ using CP-PACS data, working to chiral order $\mathcal{O}(m_\pi^3)$ and using a dipole regulator:
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![Graph showing the result for $c_0$ using CP-PACS data](image)

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\begin{itemize}
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Evidence for an Intrinsic Scale

- There is a reasonably well-defined intersection point indicating the intrinsic scale.

- For each regulator, the intersection occurs at the same value of \( \Lambda \) for both \( c_0 \) and \( c_2 \). This is a highly significant result.

- The value of the intrinsic scale differs between regulator types. The regulators are different shapes and a different cutoff is required to achieve a similar suppression of the large loop momenta.

- To obtain a systematic uncertainty in the intrinsic scale, apply a kind of \( \chi^2_{dof} \) analysis...
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- To obtain a systematic uncertainty in the intrinsic scale, apply a kind of $\chi^2_{dof}$ analysis...
On each of these renormalization flow plots, different curves correspond to different values of $m_{\pi,\text{max}}^2$.

To what extent do the curves match?

Construct $\chi^2_{dof}$, where $dof$ equals the number of $m_{\pi,\text{max}}^2$ values:

$$
\chi^2_{dof} = \frac{1}{n - 1} \sum_{i=1}^{n} \frac{(c_i(\Lambda) - c_{av}(\Lambda))^2}{(\delta c_i(\Lambda))^2},
$$

where $c_{av}(\Lambda) = \frac{\sum_{i=1}^{n} c_i(\Lambda) / (\delta c_i(\Lambda))^2}{\sum_{j=1}^{n} 1 / (\delta c_j(\Lambda))^2}$.

We shall construct $\chi^2_{dof}$ for $c_0$ and $c_2$ separately, and plot against $\Lambda$. 
On each of these renormalization flow plots, different curves correspond to different values of $m_{\pi,\text{max}}^2$.

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We shall construct $\chi^2_{dof}$ for $c_0$ and $c_2$ seperately, and plot against $\Lambda$. 
Statistical Uncertainty

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---

Statistical Uncertainty

Overview

Introduction

EFT for Nucleons

Pseudodata

Intrinsic Scale

Quenched $\rho$ Meson

Conclusion
Statistical Uncertainty

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- To what extent do the curves match?

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\chi^2_{\text{dof}} = \frac{1}{n - 1} \sum_{i=1}^{n} \left( \frac{c_i(\Lambda) - c_{\text{av}}(\Lambda)}{(\delta c_i(\Lambda))^2} \right)^2,
$$

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We shall construct $\chi^2_{\text{dof}}$ for $c_0$ and $c_2$ separately, and plot against $\Lambda$. 
Example plot: here is the result for $\chi^2_{dof}$ obtained from $c_0$ using PACS-CS data, working to chiral order $O(m_\pi^3)$ and using a dipole regulator:
The central values of \( \Lambda (\text{GeV}) \) are tabulated below:

<table>
<thead>
<tr>
<th>optimal scale</th>
<th>dipole</th>
<th>double</th>
<th>triple</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda^{\text{scale}}_{c_0, \text{JLQCD}} )</td>
<td>1.44</td>
<td>1.08</td>
<td>0.96</td>
</tr>
<tr>
<td>( \Lambda^{\text{scale}}_{c_2, \text{JLQCD}} )</td>
<td>1.40</td>
<td>1.05</td>
<td>0.94</td>
</tr>
<tr>
<td>( \Lambda^{\text{scale}}_{c_0, \text{PACS\text{--CS}}} )</td>
<td>1.21</td>
<td>0.93</td>
<td>0.83</td>
</tr>
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<td>0.93</td>
<td>0.83</td>
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<td>( \Lambda^{\text{scale}}_{c_0, \text{CP\text{--PACS}}} )</td>
<td>1.20</td>
<td>0.98</td>
<td>0.88</td>
</tr>
<tr>
<td>( \Lambda^{\text{scale}}_{c_2, \text{CP\text{--PACS}}} )</td>
<td>1.19</td>
<td>0.97</td>
<td>0.87</td>
</tr>
</tbody>
</table>
Higher Chiral Order

- We found strong scheme-dependence when working to chiral order $\mathcal{O}(m_\pi^3)$ outside the PCR.

- What happens if we try the higher chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$?
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Here is the result for $c_0$ using PACS-CS data, working to chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$ and using a dipole regulator:
Here is the result for $c_2$ using PACS-CS data, working to chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$ and using a dipole regulator:
Higher Chiral Order

Here is the result for $c_0$ using CP-PACS data, working to chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$ and using a dipole regulator:

![Graph showing the result for $c_0$ using CP-PACS data, working to chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$ and using a dipole regulator. The graph includes multiple lines with different markers, each representing a different value of $m_{\pi,\text{max}}^2$. The specific values include $m_{\pi,\text{max}}^2 = 0.54$ GeV$^2$, $m_{\pi,\text{max}}^2 = 0.69$ GeV$^2$, $m_{\pi,\text{max}}^2 = 0.70$ GeV$^2$, $m_{\pi,\text{max}}^2 = 0.91$ GeV$^2$, and $m_{\pi,\text{max}}^2 = 0.94$ GeV$^2$. The x-axis represents $\Lambda$ (GeV), ranging from 0.0 to 2.4 GeV, and the y-axis represents $c_0$ ranging from 0.6 to 1.2.]}
Here is the result for $c_2$ using CP-PACS data, working to chiral order $\mathcal{O}(m_\pi^4 \log m_\pi)$ and using a dipole regulator:

\[ m_{\pi,\text{max}}^2 = \begin{array}{ll}
0.54 \text{ GeV}^2 \\
0.69 \text{ GeV}^2 \\
0.70 \text{ GeV}^2 \\
0.91 \text{ GeV}^2 \\
0.94 \text{ GeV}^2 
\end{array} \]
At higher chiral order, there are no clear intersection points. We are unable to identify an intrinsic scale.

This means that the scheme-dependence is weakened by working to higher chiral order.

This systematic error in $c_0$ and $c_2$ is larger than their statistical errors, thus indicating that the data is outside the PCR.

There are now at least two ways of assessing the systematic uncertainty in $\Lambda$:

- from the $\chi^2_{dof}$ analysis at $O(m_\pi^3)$,
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  - from the $\chi^2_{dof}$ analysis at $O(m_\pi^3)$,
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We are now able to extrapolate $M_{N,phys}$ and obtain $c_0$ and $c_2$ by using FRR $\chi$EFT and selecting the intrinsic scale.

We are also able to provide a realistic systematic error in the result.

Examples using the dipole regulator, with uncertainties (stat)(sys- # of points)(sys- $\Lambda$):

- $c_0^{PACS-CS} = 0.900(51)(15)(6)$ (GeV),
- $c_2^{PACS-CS} = 3.06(32)(15)(25)$ (GeV$^{-1}$),
- $M_{N,phys}^{PACS-CS} = 0.967(45)(43)(3)$ (GeV).

re: PACS-CS data uses non-perturbatively $O(a)$-improved Wilson quark action at $L = 2.9$ fm.
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‘The Challenge’

- **Consider the quenched $\rho$ meson.**
  - We want to **predict** the mass of the quenched $\rho$ meson at physical pion mass ($m_{\pi,\text{phys}} = 140$ MeV).
  - We have **quenched lattice QCD (QQCD)** results from the Kentucky Group, but we are blinded to the lowest energy data.
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- Consider the quenched $\rho$ meson.

- We want to predict the mass of the quenched $\rho$ meson at physical pion mass ($m_{\pi,\text{phys}} = 140$ MeV).

- We have quenched lattice QCD (QQCD) results from the Kentucky Group, but we are blinded to the lowest energy data.

- QQCD observables are an important testing ground, since there are no experimentally known values that can introduce a prejudice in the final result.
The following data from Kentucky Group \((L = 3.06 \text{ fm})\) are missing points close to the chiral limit \((m_q = 0)\).

- The available data lie in the range \(380 < m_\pi < 720 \text{ MeV}\),
- The unavailable data lie in the range \(200 < m_\pi < 380 \text{ MeV}\).
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QQCD Data from the Lattice

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![Graph showing $m_\rho$ vs $m_\pi^2$]
The quenched \( \rho \) meson mass \( m_{\rho,Q} \) has a \textbf{similar chiral expansion to the nucleon}.

- The expansion similarly contains a \textbf{residual series and loop integrals}. We will work to chiral order \( \mathcal{O}(m_{\pi}^4) \).
- The renormalization of the low energy constants takes place just as before. The fit parameters are \( c_0, c_2 \) and \( c_4 \).
- We can generate some pseudodata as before, and plot some renormalization flow curves.
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Firstly, try pseudodata created at $\Lambda^c_\theta = 0.5$ GeV using a step function regulator ($u^2(k; \Lambda) = \theta(\Lambda - k)$).

Analyze $c_0$: 

![Graph showing $c^0$ as a function of $\Lambda$]
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Analyze $c_0$: 

![Graph showing $c_0$ as a function of $\Lambda$ with different $m^2_{\pi,max}$ values.](image-url)
Finite-Range Regulators

- Analyze $c_2$:

![Graph showing $c_2$ vs $\Lambda$ (GeV)]

- Analyze $m_{\pi,max}^2$:
  - $m_{\pi,max}^2 = 0.04$
  - $m_{\pi,max}^2 = 0.25$
  - $m_{\pi,max}^2 = 0.5$
Finite-Range Regulators

Analyze $c_4$. Notice the chiral series truncation effect.
Finite-Range Regulators

- Now let's check to see if results are regulator independent.
- Consider pseudodata created using the dipole regulator, with $\Lambda_c = 0.8$ GeV. Analyze $c_0$:
Finite-Range Regulators

Analyse $c_2$:

![Graph showing analysis of $c_2$ vs. $\Lambda$ (GeV) for different $m_{\pi,\text{max}}$ values.](image)
\( c_4 \) is also problematic.
The dipole regulator renormalization procedure was unsuccessful.

There are scheme-dependent extra non-analytic terms in the chiral expansion that have not been provided for in the fit. Pulling out the explicit $\Lambda$-dependence:

$$\tilde{\Sigma}_{Q}^{\eta',\eta'} = \chi_{\eta',\eta'} m_{\pi} + \frac{b_{3}^{\eta',\eta'}}{\Lambda^{2}} m_{\pi}^{3} + \frac{b_{5}^{\eta',\eta'}}{\Lambda^{4}} m_{\pi}^{5} + O(m_{\pi}^{6}),$$

$$\tilde{\Sigma}_{Q}^{\eta'} = \chi_{\eta'} m_{\pi}^{3} + \frac{b_{5}^{\eta'}}{\Lambda^{2}} m_{\pi}^{5} + O(m_{\pi}^{6}).$$
Test for an Intrinsic Scale

- **Choice:** We could use an $a_3$ and an $a_5$ parameter to contain the contribution from these terms, or:

- **Better:** choose a regulator which eliminates these extra terms to finite order.

- The **triple dipole regulator** is sufficient to suppress the $m_\pi^{3,5}$ terms.

- We shall use it exclusively from now on.
Test for an Intrinsic Scale

- Here is the result for $c_0$ using Kentucky Group data, working to chiral order $\mathcal{O}(m_\pi^4)$ and using a triple dipole regulator:

![Graph showing $c_0$ versus $\Lambda$ (GeV)]
Here is the result for $c_2$ using Kentucky Group data, working to chiral order $\mathcal{O}(m_\pi^4)$ and using a triple dipole regulator:
Test for an Intrinsic Scale

- Here is the result for $c_4$ using Kentucky Group data, working to chiral order $\mathcal{O}(m_\pi^4)$ and using a triple dipole regulator:
Test for an Intrinsic Scale

- The crossings are much **harder to identify**, so we will rely on our $\chi^2_{dof}$ method.
- Here is the result for $\chi^2_{dof}$ obtained from the same $c_0$: 

![Graph showing $\chi^2_{dof}$ vs $\Lambda$ (GeV)]
Test for an Intrinsic Scale

- The crossings are much harder to identify, so we will rely on our $\chi^2_{dof}$ method.
- Here is the result for $\chi^2_{dof}$ obtained from the same $c_0$:
The central, upper and lower values of $\Lambda$ (GeV) are tabulated below:

<table>
<thead>
<tr>
<th>scale (GeV)</th>
<th>for $c_0$</th>
<th>for $c_2$</th>
<th>for $c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_{\text{central}}$</td>
<td>0.64</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>$\Lambda_{\text{upper}}$</td>
<td>0.76</td>
<td>0.70</td>
<td>0.68</td>
</tr>
<tr>
<td>$\Lambda_{\text{lower}}$</td>
<td>0.52</td>
<td>0.58</td>
<td>0.59</td>
</tr>
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By averaging the result for the central value, the upper and the lower limits among $c_0$, $c_2$, and $c_4$, the optimal regulator scale $\Lambda_{\text{trip}}^{\text{scale}}$ for the quenched $\rho$ meson mass can be calculated for this data set.

Using the triple dipole regulator, $\Lambda_{\text{trip}}^{\text{scale}} = 0.64$ GeV $(+0.08 - 0.07)$ GeV.
The Intrinsic Scale

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Completing ’The Challenge’

- The extrapolation of the quenched $\rho$ meson mass can now be completed.
  - Treating the various coupling constants and $\Lambda_{\text{scale}}$ trip as independent, their errors can be added in quadrature.
  - We shall plot an inner error bar corresponding to the systematic error coming from the choice in parameters only.
  - We shall plot an outer error bar corresponding to the systematic and statistical errors of each point added in quadrature.
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- Here is the result of the extrapolation, filling in for the missing Kentucky Group data points.
- At the physical point, we find \( m_{\rho,Q}(m_{\pi,\text{phys}}^2) = 0.915 \text{ GeV} (\pm 0.036) \text{ GeV} \), an error just under 4%.
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Now, the lattice results are added to the plot:
Completing 'The Challenge'

- Here, the error bars are correlated relative to the lightest data point in the original set, \( m_{\pi}^2 = 0.143 \text{ GeV}^2 \).
- Our extrapolation error bars are smaller than for the numerically evaluated data.
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Conclusion

- We have been able to **extrapolate current lattice QCD results to the physical point**, using **Chiral Effective Field Theory**.

- We have discovered that **Finite-Range Regularization** is instrumental for the analysis of data extending outside the **chiral Power Counting Regime**.

- We have developed a **robust procedure for quantifying the degree of scheme-dependence**, through the search for an **intrinsic scale** $\Lambda^{\text{scale}}$.

- In quenched QCD, we have shown that the extrapolation scheme can make **quantifiable predictions** without phenomenologically motivated assumptions.
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Future Directions

- An alternative technique for propagation of uncertainty in the scale-dependence would be to consider marginalization of the scale.
- The extrapolation scheme can be applied to other observables such as magnetic moment and charge radii of octet baryons, which have large chiral curvature.
- Finite volume corrections are of particular interest when considering such observables.
- The extrapolation scheme will also be useful for calculating the Roper resonance, which is difficult to evaluate in lattice QCD.
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